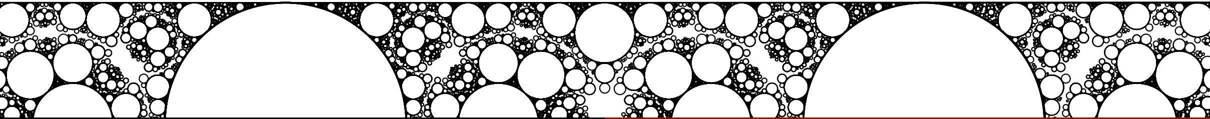
Sphere packings, rational curves, and Coxeter graphs

Arthur Baragar

University of Nevada, Las Vegas

October 24th, 2019



Think geometrically, act algebraically.

-John Tate

The Markoff equation

$$\mathcal{M}_3 : \qquad x^2 + y^2 + z^2 = 3xyz$$

- Studied by Markoff (1881) because of its relation to Diophantine approximation.
- Has three obvious automorphisms, the Viète involutions $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_3 : \qquad (x, y, z) \mapsto (x, y, 3xy - z)$$

- Starting with $(1, 1, 1)$, this gives a tree of integer solutions.

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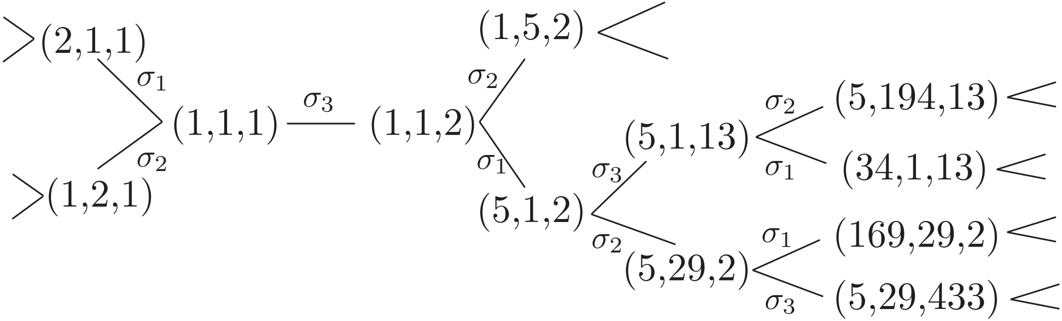

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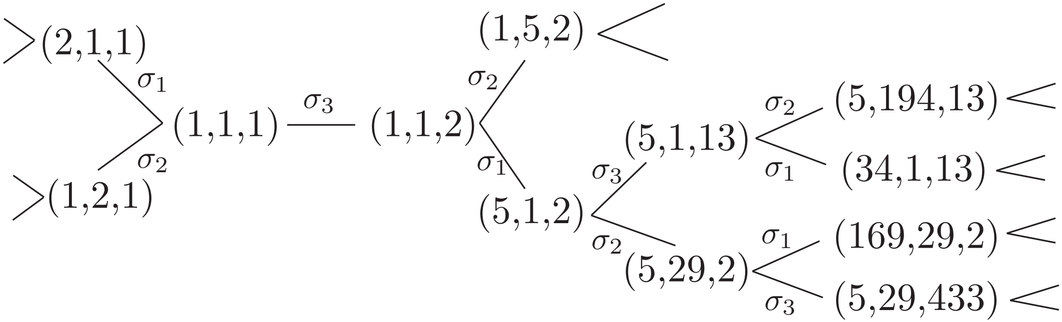

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Theorem (Zagier, '82)

$$N_{\mathcal{M}_3(\mathbb{Z}^+)}(B) = \#\{(x,y,z) \in \mathcal{M}_3(\mathbb{Z}^+) : \max\{x,y,z\} < B\}$$
$$= c(\log B)^2 + O(\log B(\log\log B)^2)$$

(From Zagier, 1982)

- The map Ψ is node to node. It is approximately logarithmic.
- The nodes in the *Euclid tree* $\mathfrak{E}$ are coprime pairs.

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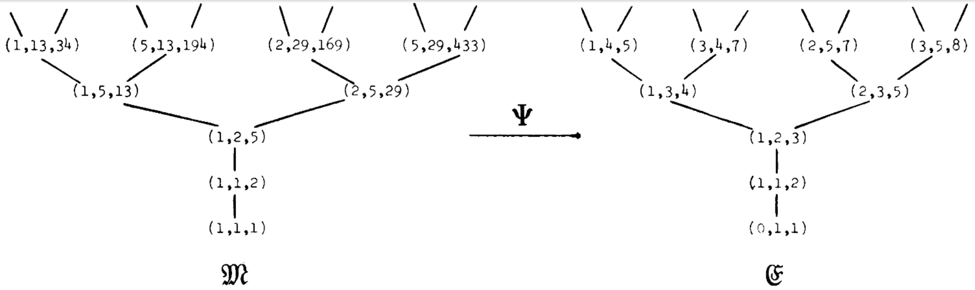

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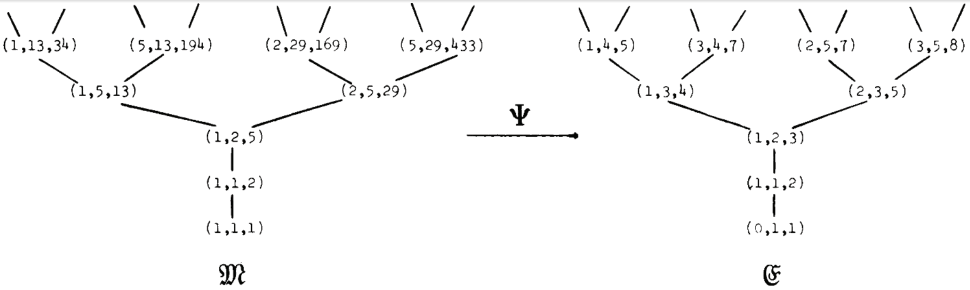

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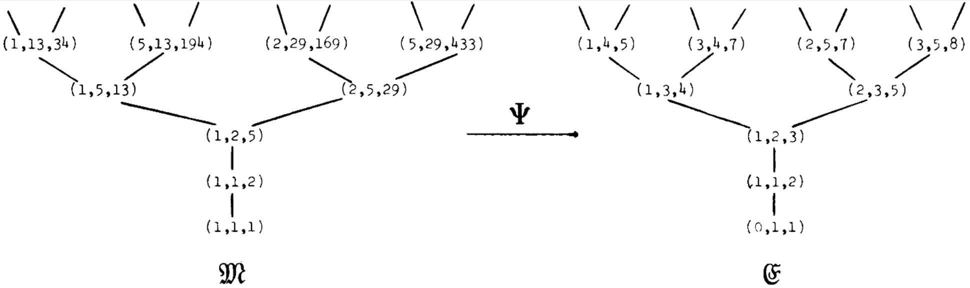

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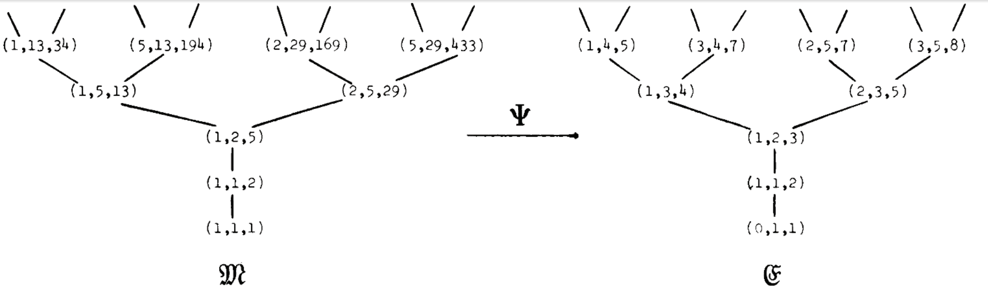

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For $\mathcal{M}_4: \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4x_1x_2x_3x_4,$

$$N_{\mathcal{M}_4(\mathbb{Z}^+)}(B) = c(\log B)^\beta + o((\log B)^\beta),$$

where $2.43 < \beta < 2.477$.

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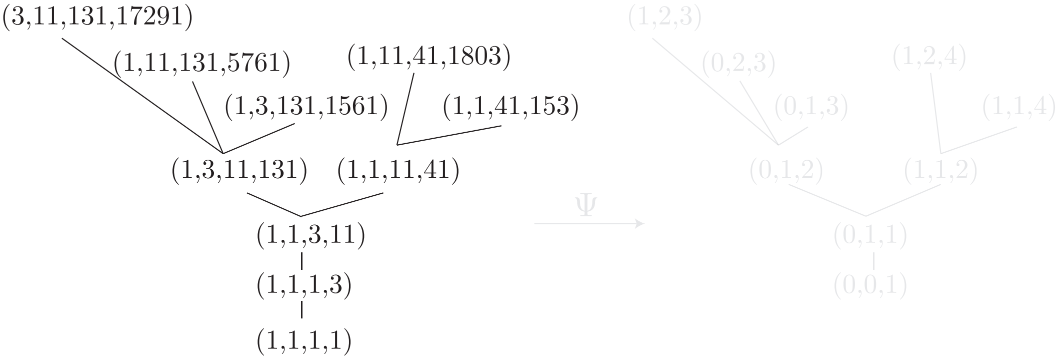

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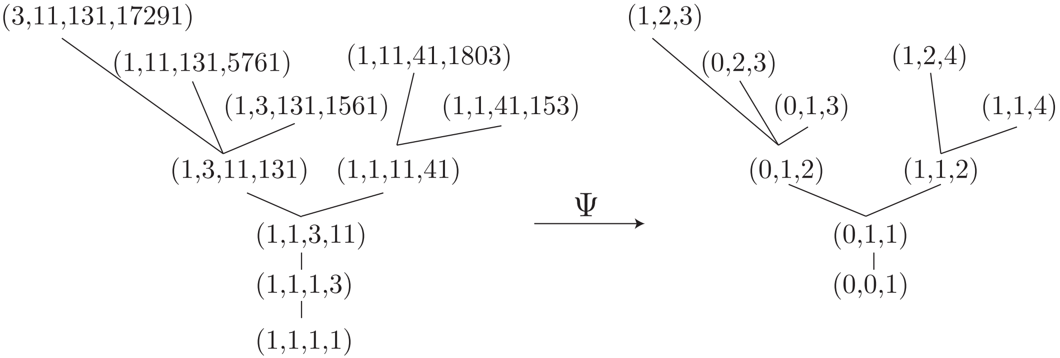
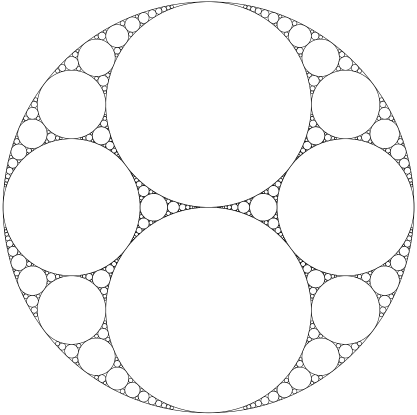


The Apollonian packing



Theorem (Boyd, '82; Kontorovich and Oh, '11; Lee and Oh, '13)

Let $\mathcal{A}$ be the set of curvatures in an Apollonian packing. Then there exists a $\mu > 0$ so that

$$N_{\mathcal{A}}(B) = cB^{\alpha} + O(B^{\alpha-\mu})$$

where $1.30 < \alpha < 1.315$ is the Huasdorff dimension of the residual set.

The exponent β for the Hurwitz equation looks like a fractal dimension. Is there a picture?

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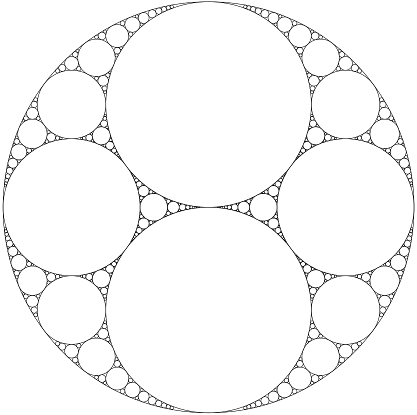

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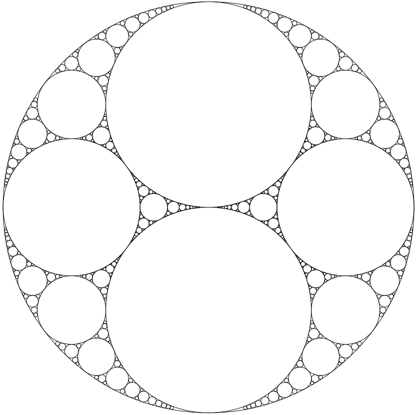

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Smooth $(2, 2, 2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

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- Similar $\mathcal{G} = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \leq \mathrm{Aut}(\mathcal{X})$.
- A K3 surface. (Wehler, '88)

Theorem (B., '96)

For $\mathcal{X}$ generic of this type, and P generic in $\mathcal{X}$,

$$N_{\mathcal{G}(P)}(B) \gg\ll \log B.$$

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Generic case

$$\mathcal{G}(P) \quad \xrightarrow{\Psi}$$

The right hand side is the classical Gauss lattice point problem in hyperbolic geometry. Asymptotics are due to Patterson, '75.

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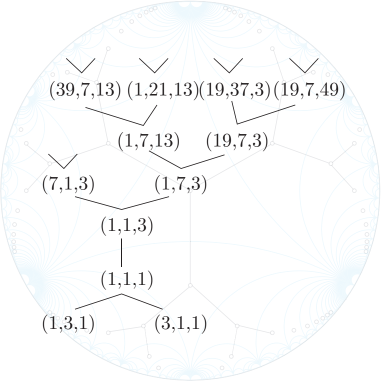

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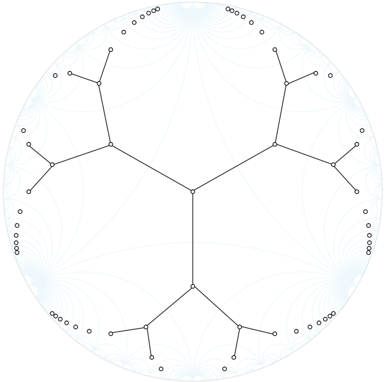

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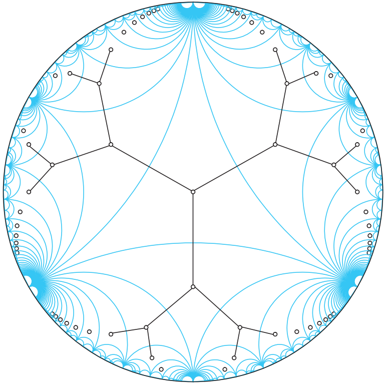

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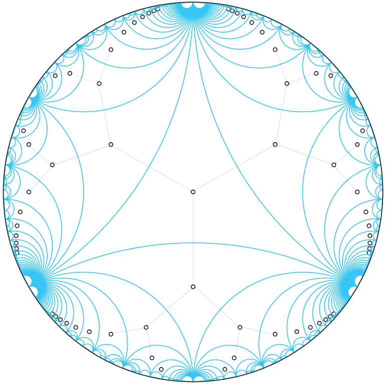

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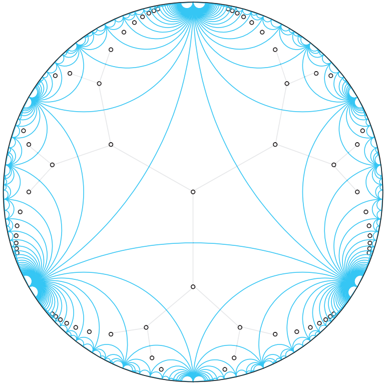

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A non-generic case

$$\mathcal{X}: \qquad F(X,Y,Z) = F_0(Y,Z)X_0^2 + F_1(Y,Z)X_0X_1 + F_2(Y,Z)X_1^2 = 0$$

- In the generic case, the curves $F_i(Y,Z)$ do not intersect in $\mathbb{P}^1 \times \mathbb{P}^1$.
- What if there exists a point $Q = (Q_y, Q_z) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $F_i(Q_y, Q_z) = 0$ for all i? (... but is otherwise generic.)
- Then $\mathcal{X}$ includes the line

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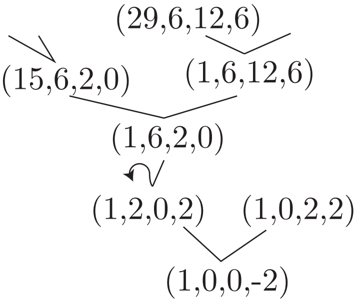

The map Ψ is $(\deg_T(X), \deg_T(Y), \deg_T(Z)$, intersection with $L)$... passage to the Picard group. Note $[\mathcal{A} : \mathcal{G}] = \infty$, where $\mathcal{A} = \mathrm{Aut}(\mathcal{X})$...

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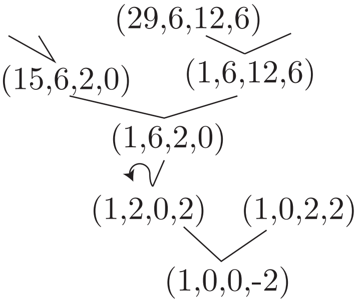

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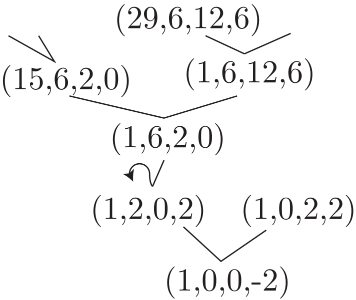

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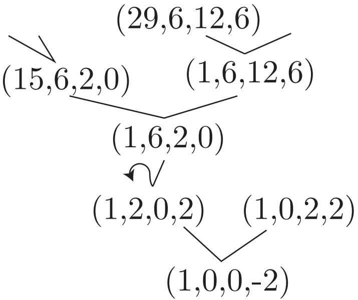

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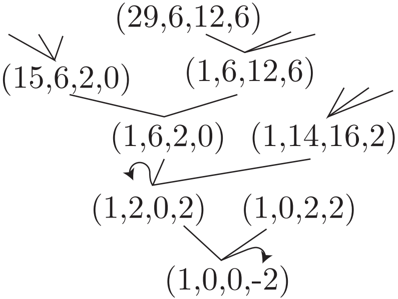

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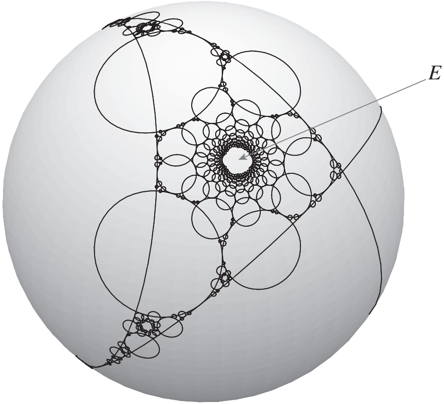

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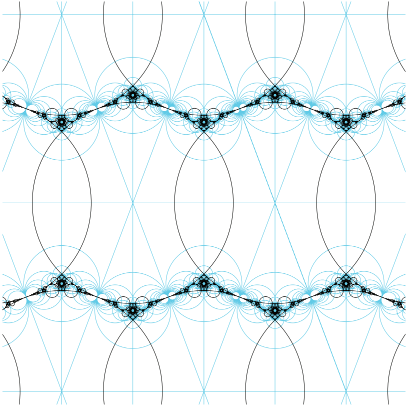

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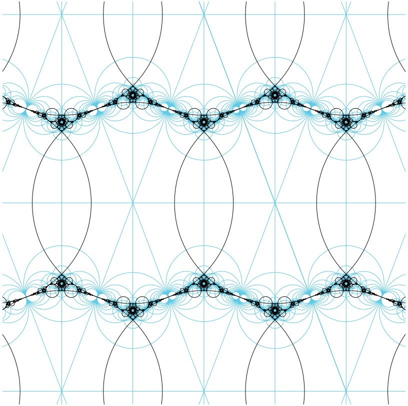

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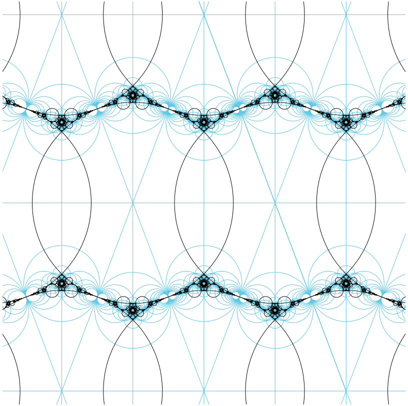

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where δ is the Hausdorff dimension of the residual set (using Oh et al.).

$\delta \sim 1.29$ (experimental).

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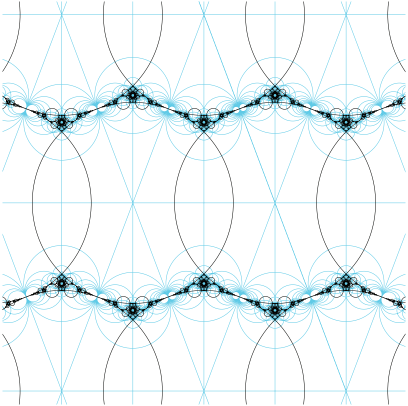

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The Apollonian circle packing

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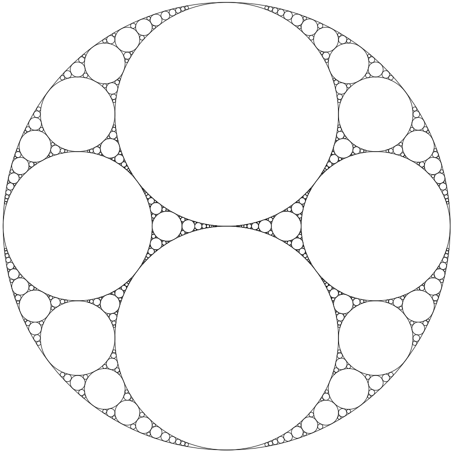
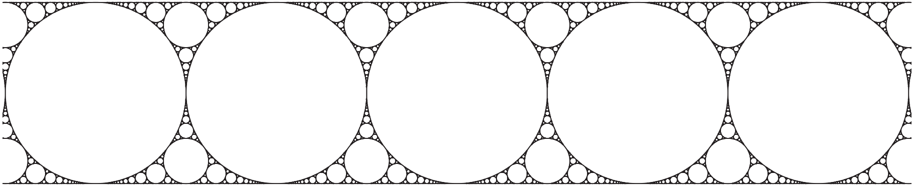


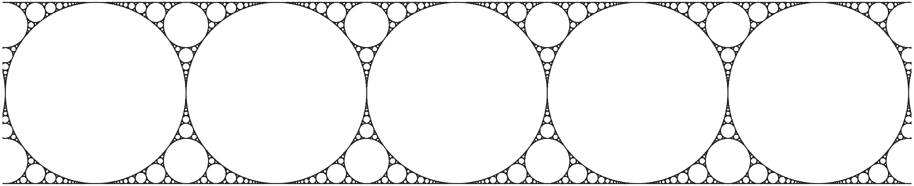


Theorem (B., '17)

There exists a K3 surface $\mathcal{X}$ such that the configuration of circles generated by its smooth rational curves is exactly the Apollonian circle packing.

$$[\mathbf{e}_i \cdot \mathbf{e}_j] = \begin{bmatrix} -2 & 2 & 2 & 2 \\ 2 & -2 & 2 & 2 \\ 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & -2 \end{bmatrix}$$

... uses a result by Morrison ('84).



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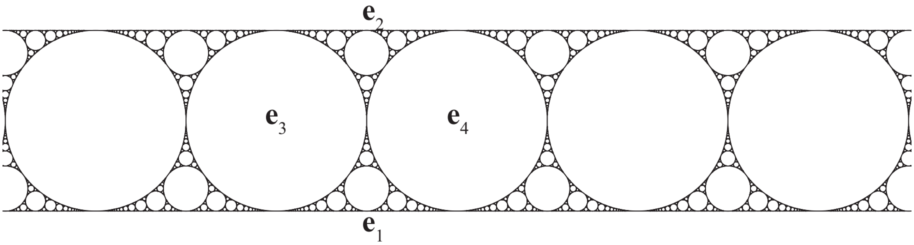

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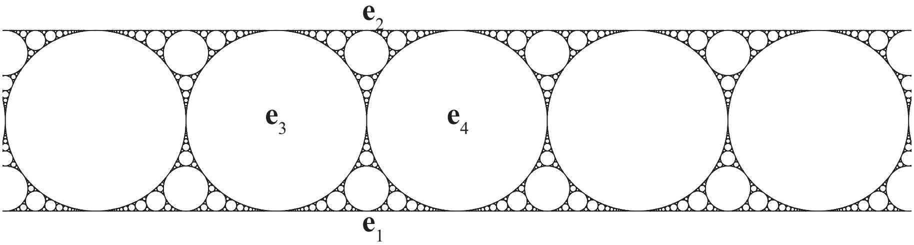

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The Soddy sphere packing

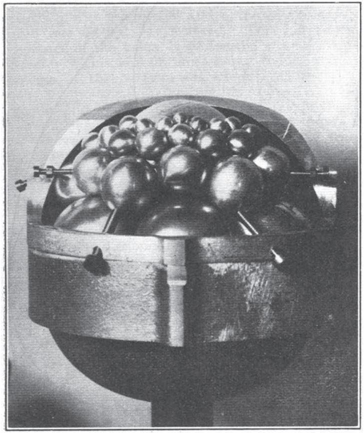


Fig. 2. (Soddy, '37)

$$[\mathbf{e}_i \cdot \mathbf{e}_j] = \begin{bmatrix} -2 & 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & 2 & 2 \\ 2 & 2 & -2 & 2 & 2 \\ 2 & 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & 2 & -2 \end{bmatrix}$$

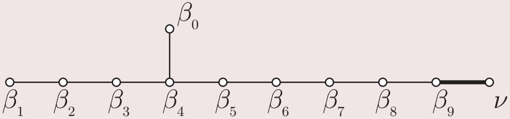
Excerpt of *Mathematical Review* MR0350626 (50 # 3118) for Boyd's paper ('74)

"… and he uses these conditions to exhibit a total of thirteen infinite packings in dimensions 2, 3, 4, 5 and 9. These examples include the Apollonian (2.1) and Soddy (3.1) packings that arise from any cluster of mutually tangent balls in dimension 2 or 3, respectively. The other examples are particularly intriguing because mutually tangent clusters do not give rise to packings in dimension $n \geq 4$."

Generic nodal Enriques surfaces

Theorem (Coble, 1919; Looijenga; Cossec and Dolgachev, '89; Allcock, '18)

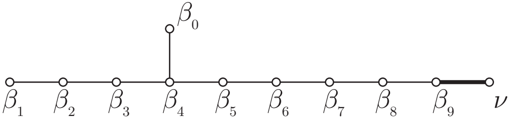
Suppose X is a generic nodal Enriques surface with nodal curve ν. Let Λ be its Picard group modulo torsion. Then there exist $\beta_0, \ldots, \beta_9 \in \Lambda$ so that $\beta_i \cdot \beta_i = -2$ and $\beta_1, \ldots, \beta_9, \nu$ are the nodes of the Coxeter graph:



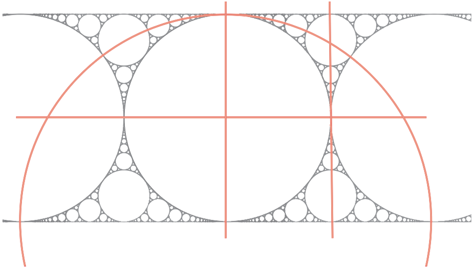
Let

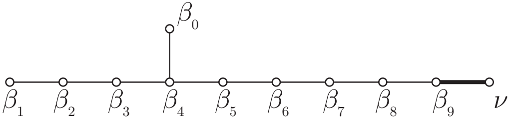
$$\Gamma = \langle R_{\beta_0}, \ldots, R_{\beta_9} \rangle \cong W_{246}.$$

Then the image in Λ of all nodal curves on X is the Γ-orbit of ν.



- Each node represents a vector/plane.
- No edge means they are perpendicular: $\beta_i \cdot \beta_j = 0$.
- A regular edge means the vectors are at an angle of $2\pi/3$: $\beta_i \cdot \beta_j = 1$ ($\beta_i^2 = -2$).
- A bold edge means the vectors are parallel: $\beta_9 \cdot \nu = 2$.





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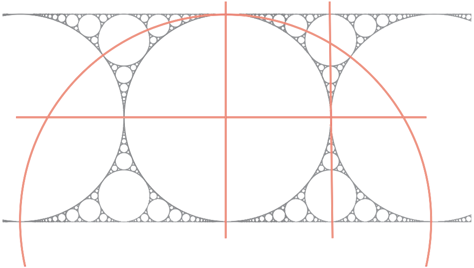

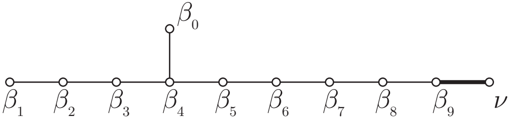

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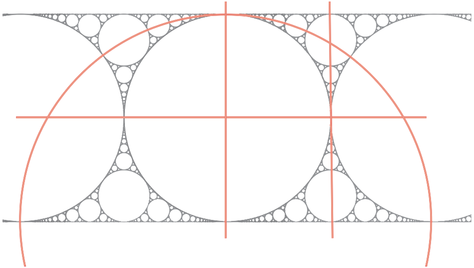

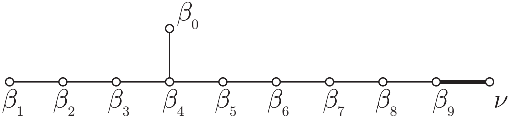

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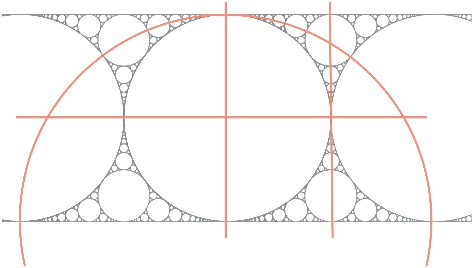

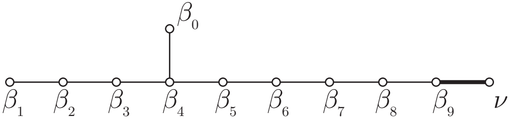

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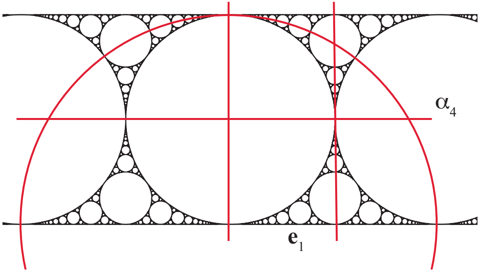

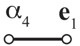

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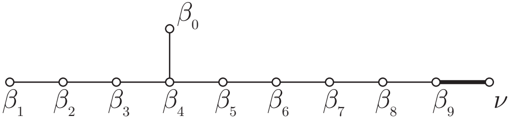

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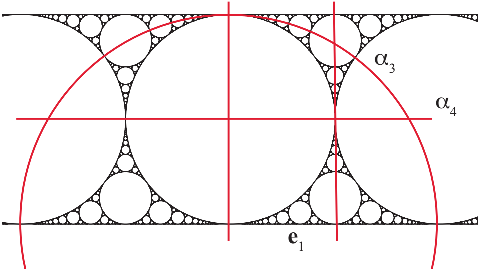

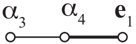

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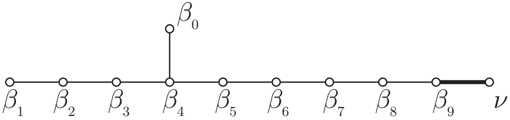

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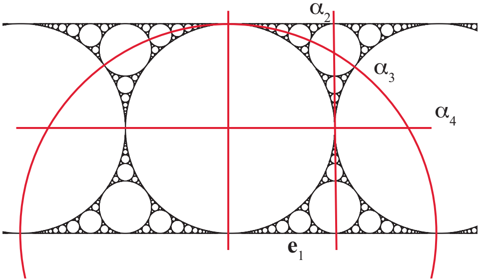

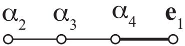

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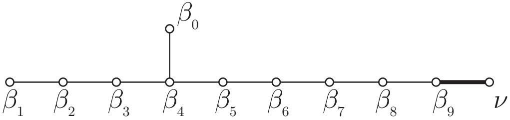

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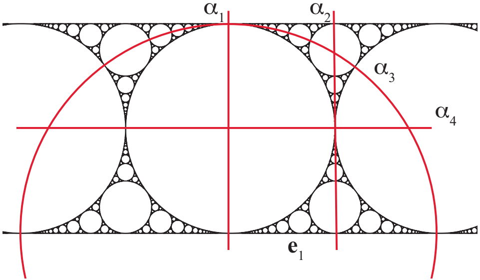

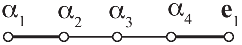

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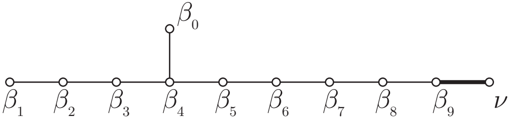

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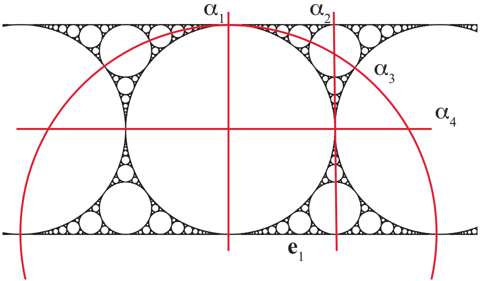

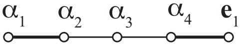

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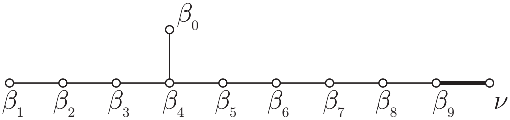

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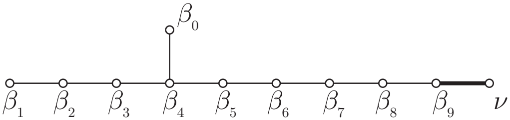

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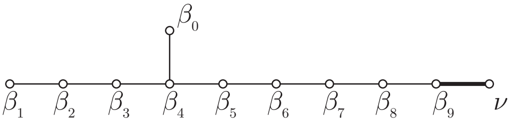
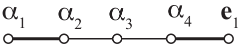
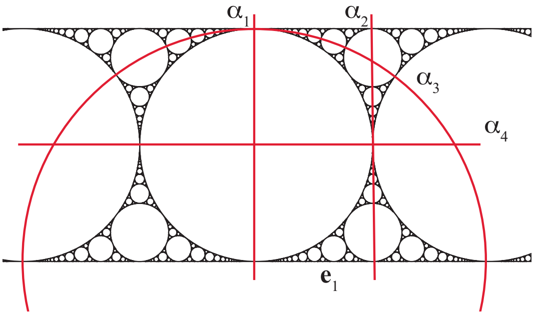
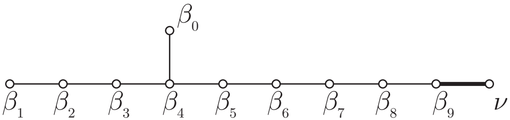
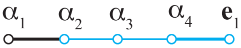




$$R_{\beta_i}(\mathbf{x}) = \mathbf{x} - 2\text{proj}_{\beta_i}(\mathbf{x})$$
$$= \mathbf{x} - 2\frac{\mathbf{x}\cdot\beta_i}{\beta_i\cdot\beta_i}\beta_i$$
$$= \mathbf{x} + (\mathbf{x}\cdot\beta_i)\beta_i.$$
$$\mathbf{e}_0 = \nu$$
$$\mathbf{e}_1 = R_{\beta_9}(\mathbf{e}_0)$$
$$\mathbf{e}_{i+1} = R_{\beta_{10-i}}(\mathbf{e}_i)$$







$$
\begin{aligned}
R_{\beta_i}(\mathbf{x}) &= \mathbf{x} - 2\mathrm{proj}_{\beta_i}(\mathbf{x}) \\
&= \mathbf{x} - 2\frac{\mathbf{x}\cdot\beta_i}{\beta_i\cdot\beta_i}\beta_i \\
&= \mathbf{x} + (\mathbf{x}\cdot\beta_i)\beta_i .
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{e}_0 &= \nu \\
\mathbf{e}_1 &= R_{\beta_9}(\mathbf{e}_0) \\
\mathbf{e}_{i+1} &= R_{\beta_{10-i}}(\mathbf{e}_i)
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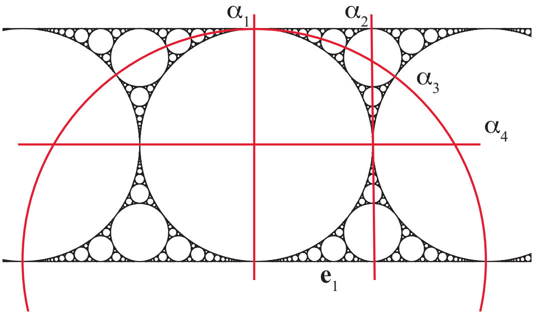

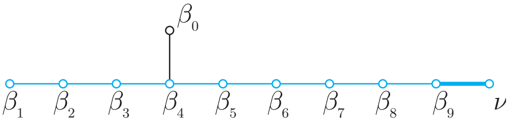

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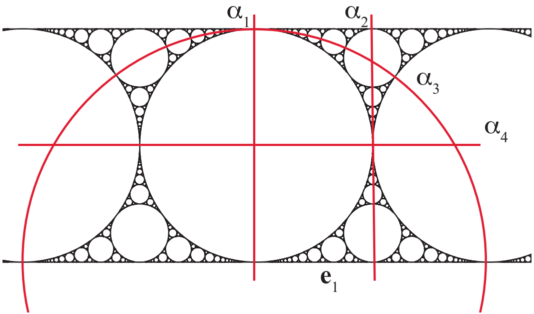

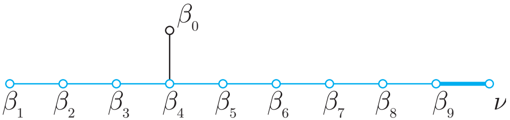

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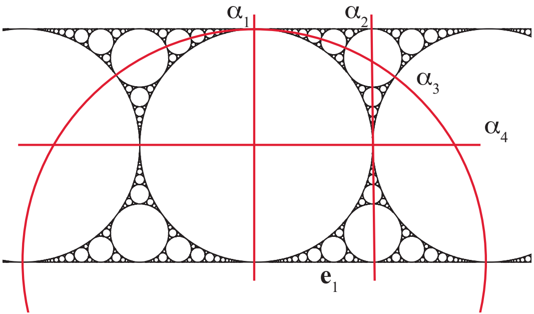

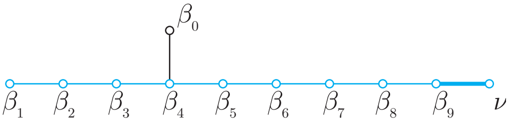

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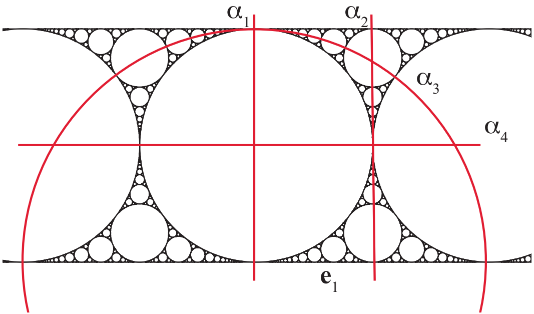


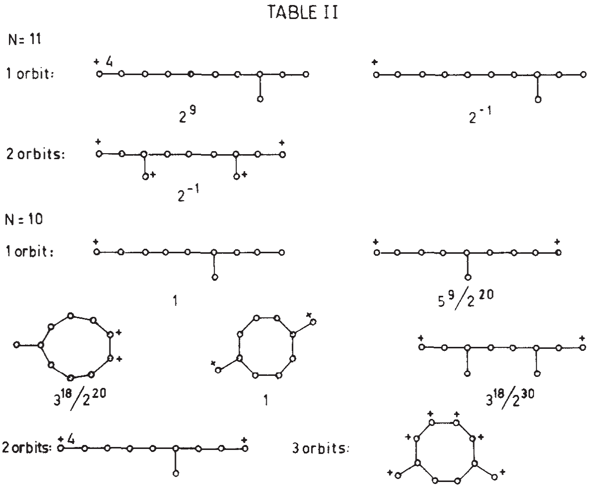
$$[\mathbf{e}_i \cdot \mathbf{e}_j] = \begin{bmatrix} -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Theorem (B., '19)

The set of nodal curves on a generic nodal Enriques surface generates an Apollonian packing in eight dimensions.

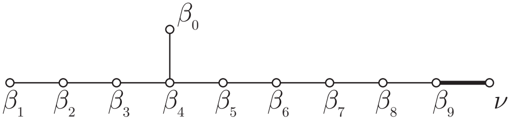
$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -2 \end{bmatrix}$$

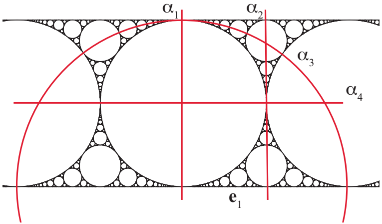


From Maxwell, ’81.

$$2[\beta_i \cdot \beta_j]^{-1} = \begin{bmatrix} 5 & 3 & 6 & 9 & 12 & 10 & 8 & 6 & 4 & 2 \\ 3 & 0 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 2 & 4 & 8 & 12 & 10 & 8 & 6 & 4 & 2 \\ 9 & 4 & 8 & 12 & 18 & 15 & 12 & 9 & 6 & 3 \\ 12 & 6 & 12 & 18 & 24 & 20 & 16 & 12 & 8 & 4 \\ 10 & 5 & 10 & 15 & 20 & 15 & 12 & 9 & 6 & 3 \\ 8 & 4 & 8 & 12 & 16 & 12 & 8 & 6 & 4 & 2 \\ 6 & 3 & 6 & 9 & 12 & 9 & 6 & 3 & 2 & 1 \\ 4 & 2 & 4 & 6 & 8 & 6 & 4 & 2 & 0 & 0 \\ 2 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & -1 \end{bmatrix}$$

$$2[\alpha_i \cdot \alpha_j]^{-1} = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$





$$2[\beta_i\cdot\beta_j]^{-1}=\begin{bmatrix}5&3&6&9&12&10&8&6&4&2\\3&0&2&4&6&5&4&3&2&1\\6&2&4&8&12&10&8&6&4&2\\9&4&8&12&18&15&12&9&6&3\\12&6&12&18&24&20&16&12&8&4\\10&5&10&15&20&15&12&9&6&3\\8&4&8&12&16&12&8&6&4&2\\6&3&6&9&12&9&6&3&2&1\\4&2&4&6&8&6&4&2&0&0\\2&1&2&3&4&3&2&1&0&-1\end{bmatrix}$$

$$2[\alpha_i\cdot\alpha_j]^{-1}=\begin{bmatrix}2&3&2&1\\3&3&2&1\\2&2&0&0\\1&1&0&-1\end{bmatrix}$$